Problem # 313

Determine all real numbers x for which $x^n + \frac{1}{x^n}$ is an integer for all positive integers n.

Solution:

Answer: $x = \frac{m \pm \sqrt{m^2 - 4}}{2}$, where m is an integer and $|m| \geq 2$.

Proof. It is easily seen that $x = 1$, $x = -1$ works for all n. If $x + \frac{1}{x}$ is an integer, then

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2 \quad \text{is an integer.}$$

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) \quad \text{is an integer.}$$

By induction if $x^k + \frac{1}{x^k}$ is an integer for $k \leq n$, then

$$x^{n+1} + \frac{1}{x^{n+1}} = \left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$

is an integer. Thus we must have $x + \frac{1}{x} = m$ where m is an integer. Then, $x^2 - mx + 1 = 0$

$$x = \frac{m \pm \sqrt{m^2 - 4}}{2} \tag{1}$$

with $|m| \geq 2$. □

Source: Suggested by Dr. T. Smotzer.