Problem # 312

Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, $n/5$ is a perfect fith power and $n/7$ is a perfect seventh power.

Solution:

Answer: $\boxed{n = 2^{105}\, 3^{70}\, 5^{126}\, 7^{120}}$

Proof. The solution must have the form $n = 2^a\, 3^b\, 5^c\, 7^d$ since the smallest positive integer solution will have only the prime factors 2, 3, 5 and 7. Notice that $\frac{n}{2}$ is a perfect square provided that 2 divides each exponent $a-1$, b, c and d. Similarly 3 divides each of a, $b-1$, c and d; 5 divides each of a, b, $c-1$ and d and 7 divides each of a, b, c and $d-1$. To find the smallest positive value of n satisfying these criteria, a must be the least common multiple of 3, 5, and 7 which is one more than a multiple of 2; i.e. $a = 3 \cdot 5 \cdot 7 = 105$. Similarly b must be the least common multiple of 2, 5, and 7 which is one more than a multiple of 3; i.e $b = 2 \cdot 5 \cdot 7 = 70$. Then c must be the least common multiple of 2, 3, and 7 which is one more than a multiple of 5; i.e., $c = (2 \cdot 3 \cdot 7) \cdot 3 = 126$. Finally d must be the least common multiple of 2, 3, and 5 which is one more than a multiple of 7;
$d = (2 \cdot 3 \cdot 5) \cdot 2^2 = 120$. □

Source: The Pentagon, Fall 2003.