Problem # 311

Let BC be a fixed line segment, ℓ a line parallel to BC, and A an arbitrary point on ℓ. Describe (with proof) the path followed by the orthocenter of $\triangle ABC$ as A moves along ℓ.

Solution:

Answer: $\boxed{y = \dfrac{x(c-x)}{b}}$

Proof. Let point B be the point $(0, 0)$ and C be the point $(c, 0)$, $c \neq 0$. Let the line ℓ be the line $y = b$, $b \neq 0$. Since A lies on ℓ, it is the point (a, b) where $a \in \mathbb{R}$. Let D be the orthocenter of $\triangle ABC$. Since D must be the point where the three altitudes of the triangle ABC intersect, it must be on the line $x = a$, the line $\overline{DB}$ $\left(y = \dfrac{(c-a)x}{b}\right)$ and the line $\overline{DC}$ $\left(y = \dfrac{a(c-x)}{b}\right)$. The three lines intersect at the common point D $\left(a, \dfrac{a(c-a)}{b}\right)$. Since a is any real number the orthocenter is given by the equation $y = \dfrac{x(c-x)}{b}$ for $x \in \mathbb{R}$ which is a parabola. □

Source: The Pentagon, Vol. 59, No. 2, Spring 2000.