Problem # 310

Find all real valued functions $f(x)$ with domain $x \neq 0$ that satisfy

$$x^2 f\left(\frac{x}{2}\right) - f\left(\frac{2}{x}\right) = 1 \ .$$

Solution:

Answer: $\boxed{f(x) = \dfrac{4+x^2}{15x^2}}$

Proof. In the given equation, replace x by $4/x$ to get $16\, f\left(\dfrac{2}{x}\right) - x^2 f\left(\dfrac{x}{2}\right) = x^2$. Adding this to the given equation gives $15\, f\left(\dfrac{2}{x}\right) = x^2 + 1$. Replacing x by $2/x$ we find $f\,(x) = \dfrac{4+x^2}{15\,x^2}$. □

Source: Suggested by Dr. Thomas Smotzer.