Problem # 166

Evaluate: $S = \dfrac{1}{2\sqrt{1}+\sqrt{2}} + \dfrac{1}{3\sqrt{2}+2\sqrt{3}} + \cdots + \dfrac{1}{100\sqrt{99}+99\sqrt{100}}$

Solution:

Answer: $\boxed{\dfrac{9}{10}}$

Proof. We rewrite the sum as a telescoping sum. Observe that

$$\begin{aligned}
\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} &= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)^2 n - n^2(n+1)} \\
&= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)n(n+1-n)} \\
&= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)n} \\
&= \frac{\sqrt{n}}{n} - \frac{\sqrt{n+1}}{n+1}
\end{aligned}$$

Therefore,

$$S = \left(\frac{\sqrt{1}}{1} - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}\right) + \cdots + \left(\frac{\sqrt{99}}{99} - \frac{\sqrt{100}}{100}\right) = 1 - \frac{10}{100} = \frac{9}{10}$$

□

Source: Mathematics Contest, Slovakia, Bratislava, 2007.