

Problem # 165

Let a , b and c be real numbers and let

$$f(x) = (x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) .$$

Show that $f(x) = 0$ has a real solution.

Solution:

Proof.

Let us assume that $a \leq b \leq c$. Then $f(a) = (a - b)(a - c) \geq 0$ and $f(b) = (b - c)(b - a) \leq 0$. The intermediate value theorem requires that there is a real solution in the interval $[a, b]$.

Note that $f(c) = (c - a)(c - b) \geq 0$. Thus there is a solution in the interval $[b, c]$.

□

Source: High School Competition, Lodz, Poland, 1999.