Problem # 164

Prove that if $0 < x < 1$ and $0 < y < 1$, then $x(1-y)^2 + y(1-x)^2 < (1-xy)^2$.

Solution:

Proof.

Multiplying out both sides of the inequality, we get

$$x - 2xy + xy^2 + y - 2xy + yx^2 < 1 - 2xy + x^2y^2 \ ,$$

which is equivalent to

$$x + xy^2 + y + yx^2 < 1 + 2xy + x^2y^2 \ ,$$

and factoring both sides we obtain

$$(x+y)(1+xy) < (1+xy)^2 \ .$$

Canceling the factor $(1+xy)$ and writing all terms on the right hand side, we get the inequality,

$$0 < 1 - x - y + xy = (1-x)(1-y) \ ,$$

which is true for the supposed values of x and y.

□

Source: Polish Regional High School Olympiad 2003.